

Hamilton-Jacobi approach to pre-big bang cosmology and the problem of initial conditions

Piret Kuusk¹ and Margus Saal²

Abstract

The Hamilton-Jacobi equation for the string cosmology is solved using the gradient expansion method. The zeroth order solution is taken to be the standard pre-big bang model and the second order solution is found for the dilaton and the three-metric. It indicates that corrections generated by inhomogeneities of the seed metric are suppressed near the singularity and are growing towards the asymptotic past, but corrections generated by the dilaton inhomogeneities are growing near the singularity and are suppressed in the past. Possible influences of initial metric inhomogeneities on the pre-big bang superinflation are discussed.

Key words: pre-big bang cosmology, Hamilton-Jacobi equation, gradient expansion method

¹ Institute of Physics, University of Tartu, Riia 142, Tartu 51014, Estonia.

² University of Tartu, Tähe 4, Tartu 51010, Estonia.

E-mail: margus@hexagon.fi.tartu.ee

1 Introduction

Observations show that at present our Universe is rather homogeneous and isotropic. It is commonly believed that the present homogeneity was achieved from a chaotic (inhomogeneous) initial state by inflation. In the string cosmology [1, 2], the same can be obtained by a pre-big bang superinflation. The pre-big bang era describes a possible evolution of the Universe from the string perturbative vacuum where classical equations for gravitational and moduli fields hold until the curvature and coupling reach their maximum and both, string and quantum corrections, become crucial. It is believed (but not yet proved) that then a graceful exit transition [3, 4] to the usual Friedmann-Robertson-Walker Universe occurs, and standard cosmological picture, with some shades, works well.

Besides the graceful exit problem, there is the problem of initial conditions: it has been argued that pre-big bang initial conditions have to be fine-tuned in order to give expected results [5]. Turner and Weinberg [6] concluded that curvature terms postpone the onset of inflation and can prevent getting sufficient amount of inflation before higher-order loop and string corrections become important. Kaloper, Linde and Bousso [7] have argued that horizon and flatness problems will be solved if the Universe at the onset of inflation is exponentially large and homogeneous. Clancy et al [8] have found, that the constraints for sufficient amount of inflation in anisotropic models are stronger than in the isotropic case. Numerical calculations presented by Maharana et al [9] and Chiba [10] contain controversial results concerning the decay of initial inhomogeneities. Counterarguments for justifying the pre-big bang inflationary model have been given in [11, 12, 13, 14].

There are several papers which discuss the role of initial inhomogeneities in cosmological models (for a review see [15]). Using the gradient expansion method developed by Salopek et al [16, 17], evolution of inhomogeneities in cosmological models which contain cosmological constant and radiation field [18], massive minimally coupled scalar field [19], or perfect fluid [20] were considered. The Brans-Dicke theory with a dust and cosmological constant was investigated by Soda et al [21] using the gradient expansion of the corresponding Hamilton-Jacobi equation. The direct method to solve the Einstein equations expanded in spatial gradients is developed and applied to various models [22]. The closest to our approach is the paper by Nambu and Taruya [23] where the role of initial inhomogeneities in the conventional inflationary cosmology is discussed in the framework of long-wavelength approximation.

In the string cosmology, inhomogeneous spherically and cylindrically symmetric models are investigated by Barrow et al [24, 25] and Feinstein et al [26] using methods familiar in the general relativity. Buonanno et al [27] link the cosmological scenario with collapsing initial gravitational waves.

In the following we shall use the gradient expansion method developed by Saygili [28] for investigating the effect of small inhomogeneities and related three-curvature. We assume the standard pre-big bang cosmology to be valid as the zeroth order approximation and include inhomogeneities as small perturbations. They generate small three-curvature which may affect the onset and duration of superinflation. Although the long-wavelength approximation is valid in a region where spatial derivatives are negligible and also inhomogeneities must be constrained, it can be used for investigat-

ing the evolution trends of small inhomogeneities and deriving combined constraints on their size and the duration of inflation. We demonstrate that the initial spatial curvature is suppressed during the superinflation and therefore neglecting the spatial gradients improves in time. We also conclude that positive initial curvature supports and negative curvature depresses the effectiveness of inflation.

2 Hamilton-Jacobi equation for effective string action

Our starting point is the low energy effective action

$$I_{eff} = \frac{1}{2\lambda_s^2} \int d^4x \sqrt{-g} e^{-\phi} [{}^4\mathcal{R} + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi]. \quad (1)$$

In the Arnowitt-Deser-Misner (ADM) formalism the dynamics is encoded in the Hamiltonian constraint [28]

$$\begin{aligned} \mathcal{H}[\pi^{ij}, \pi^\phi, \gamma_{ij}, \phi] \equiv & \frac{e^\phi}{\sqrt{\gamma}} \left[\pi^{ij} \pi^{kl} \gamma_{ik} \gamma_{jl} + \frac{1}{2} (\pi^\phi)^2 + \pi \pi^\phi \right] - \\ & - \sqrt{\gamma} e^{-\phi} R - \sqrt{\gamma} e^{-\phi} \gamma^{ij} \partial_i \phi \partial_j \phi + 2\sqrt{\gamma} \Delta e^{-\phi} = 0, \end{aligned} \quad (2)$$

where R is the 3-dimensional scalar curvature of a spacelike hypersurface. The momentum constraints [28]

$$\mathcal{H}_i[\pi^{ij}, \pi^\phi, \gamma_{ij}, \phi] \equiv -2(\gamma_{ik} \pi^{kj})_{,j} + \pi^{kl} \gamma_{kl,i} + \pi^\phi \phi_{,i} = 0. \quad (3)$$

state the reparametrization invariance on spatial hypersurfaces $x^l \rightarrow x^l + \xi^l$ (diffeomorphism invariance) [29].

The equations of motion for ϕ and γ_{ij} read [28]

$$\frac{1}{N} \left(\dot{\phi} - N^i \phi_{,i} \right) = \frac{e^\phi}{\sqrt{\gamma}} (\pi^\phi + \pi), \quad (4)$$

$$-2K_{ij} \equiv \frac{1}{N} (\dot{\gamma}_{ij} - N_{i;j} - N_{j;i}) = \frac{e^\phi}{\sqrt{\gamma}} (2\pi^{kl} \gamma_{ik} \gamma_{jl} + \gamma_{ij} \pi^\phi). \quad (5)$$

Here K_{ij} is the 3-dimensional extrinsic curvature tensor (the trace of K_{ij} is a generalization of the Hubble parameter) and $\pi = \gamma_{ij} \pi^{ij}$ is the trace of the gravitational momentum tensor. Lapse function N and shift vector N^i describe the ADM 3+1 decomposition of the spacetime. In the following we use a synchronous gauge, i.e. we put $N(t, x^k) = 1$ and $N^i(t, x^k) = 0$.

The solutions of equations for the momenta can be determined from a particular solution of the corresponding Hamilton-Jacobi equation [28]

$$\begin{aligned} & \frac{e^\phi}{\sqrt{\gamma}} \left[\frac{\delta S}{\delta \gamma_{ij}} \frac{\delta S}{\delta \gamma_{kl}} \gamma_{ik} \gamma_{jl} + \frac{1}{2} \left(\frac{\delta S}{\delta \phi} \right)^2 + \gamma_{ij} \frac{\delta S}{\delta \gamma_{ij}} \frac{\delta S}{\delta \phi} \right] \\ & - \sqrt{\gamma} e^{-\phi} R - \sqrt{\gamma} e^{-\phi} \gamma^{ij} \partial_i \phi \partial_j \phi + 2\sqrt{\gamma} \Delta e^{-\phi} = 0, \end{aligned} \quad (6)$$

as

$$\pi^\phi = \frac{\delta S}{\delta \phi}, \quad \pi^{ij} = \frac{\delta S}{\delta \gamma_{ij}}. \quad (7)$$

The Hamilton-Jacobi equation is obtained by replacing momenta (7) into the Hamiltonian constraint (2).

After solving the Hamilton-Jacobi equation for S , we get an opportunity to solve also field equations (4), (5), and consequently determine the evolution of 3-metric and dilaton. Although there is no hope to find an exact solution to the Hamilton-Jacobi equation, it is possible to obtain approximate solutions.

3 Long-wavelength approximation

We use the basic formalism of gradient expansion [17] for investigating the Hamilton-Jacobi equation (6). Functional $S[\gamma_{ij}(x), \phi(x)]$ is expanded in a series of terms according to the number of spatial gradients they contain:

$$S[\gamma_{ij}(x), \phi(x)] = S^{(0)} + S^{(2)} + S^{(4)} + \dots \quad (8)$$

Here $S^{(0)}$ contains no spatial gradients, $S^{(2)}$ contains two spatial gradients, and so on. Solving the Hamilton-Jacobi equation order-by-order amounts to requiring the Hamiltonian constraint to vanish at each order. We assume that each term in expansion (8) satisfy also the momentum constraint (3).

The long-wavelength approximation is actually the assumption that the characteristic comoving coordinate scale of spatial variations L_{cm} (wavelength of inhomogeneities) is larger than the comoving Hubble radius $d_{cm} = (H a)^{-1}$, $d_{cm} \ll L_{cm}$. In terms of physical scale ($L_{ph} = a L_{cm}$), the same is written as $H^{-1} \ll L_{ph}$. On scales less than L_{ph} the time derivatives dominate over spatial gradients and space is almost homogeneous. Inside the homogeneous region the zeroth order three-metric and dilaton are time dependent, but approximately coordinate independent quantities. During the pre-big bang superinflation, the Hubble radius H^{-1} shrinks in time and therefore the efficiency of the gradient expansion grows in time, as distinct from the case of the de Sitter inflation, where the Hubble radius remains approximately constant.

In the standard pre-big bang cosmology it is assumed that the initial state was an approximately flat spacetime. However, in the asymptotic past, it is natural to assume a generic (classical) initial state. Using the gradient expansion method, we can investigate a generic initial state perturbatively, i.e. in the second approximation we incorporate the spatial gradients which represent the curvature and dilaton inhomogeneities.

4 The zeroth order solution

In the zeroth order (long-wavelength approximation), the Hamilton-Jacobi equation does not contain spatial derivatives and we can neglect the last three terms in equation (6)

$$\frac{e^\phi}{\sqrt{\gamma}} \left[\frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \gamma_{kl}} \gamma_{ik} \gamma_{jl} + \frac{1}{2} \left(\frac{\delta S^{(0)}}{\delta \phi} \right)^2 + \gamma_{ij} \frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \phi} \right] = 0. \quad (9)$$

Let its solution be in the form (cf [28])

$$S^{(0)} = \frac{4}{\sqrt{3}-1} \int e^{-\phi} H[\phi(t, x^k)] \sqrt{\gamma} d^3x. \quad (10)$$

The integral is taken over invariant three-volume $\sqrt{\gamma} d^3x$ and the functional is therefore diffeomorphism invariant. As required, $S^{(0)}$ contains no spatial gradients. The prefactor is chosen so that H corresponds to the usual Hubble parameter in the long-wavelength approximation (see Sect. 6). Upon substituting (10) into equation (9) the Hamilton-Jacobi equation in the zeroth approximation reduces to a differential equation

$$\left(\frac{\partial H}{\partial \phi} \right)^2 + H \left(\frac{\partial H}{\partial \phi} \right) - \frac{1}{2} H^2 = 0. \quad (11)$$

Direct integration yields

$$H(\phi) = H_0 e^{\left(\frac{\pm\sqrt{3}-1}{2} \right) \phi}. \quad (12)$$

Here H_0 is an integration constant which will be specified later. Since $H(\phi)$ is proportional to the usual Hubble parameter we choose the upper sign in the exponent, because this is in agreement with general principles of the standard pre-big bang model, which states that in the asymptotic past the Universe was sufficiently flat i.e. $\phi \rightarrow -\infty \Leftrightarrow H(\phi) \rightarrow 0$.

Taking into account expressions for canonical momenta (7) and for H (12), field equations (4), (5) in the zeroth order read

$$\dot{\phi}^{(0)} = \frac{2\sqrt{3}}{\sqrt{3}-1} H, \quad \dot{\gamma}_{ij}^{(0)} = 2H\gamma_{ij}. \quad (13)$$

Equation for the zeroth order dilaton can be directly integrated

$$\phi^{(0)}(t) = -\frac{2}{\sqrt{3}-1} \ln \left[\sqrt{3} H_0 (t_0 - t) \right], \quad t < t_0. \quad (14)$$

Here t_0 is an integration constant which, in general, depends on spatial coordinates x^l . But these zeroth order inhomogeneities of the dilaton field can be removed by a suitable choice of synchronous gauge [30] and in the following we take $t_0 = \text{const}$. Parameter t_0 corresponds, in our interpretation, to the moment when quantum and string effects become significant (classical singularity) and roughly to the moment when superinflation ends: $t_0 = t_{\text{singularity}} \approx t_f$. From solution (14) we also get a restrictive condition for H_0 , namely, it should be positive, $H_0 > 0$. For latter convenience, let us take $H_0 = 1/\sqrt{3}$, then the full zeroth order solution reads

$$\phi^{(0)}(t) = -\frac{2}{\sqrt{3}-1} \ln (t_0 - t), \quad t < t_0, \quad (15)$$

$$\gamma_{ij}^{(0)}(t, x^l) = (t_0 - t)^{-\frac{2}{\sqrt{3}}} k_{ij}(x^l), \quad t < t_0, \quad (16)$$

where the seed metric $k_{ij}(x^l)$ is an arbitrary function of spatial variables alone. In the zeroth order solution it is assumed that it describes the fluctuations which have wavelengths larger than the Hubble radius and inside it the geometry is approximately homogeneous and flat.

5 The second order solution

The second order Hamilton-Jacobi equation reads

$$\begin{aligned} \mathcal{H}^{(2)} &= 2 H \gamma_{ij} \frac{\delta S^{(2)}}{\delta \gamma_{ij}} + \frac{2\sqrt{3}}{\sqrt{3}-1} H \frac{\delta S^{(2)}}{\delta \phi} \\ &\quad - \sqrt{\gamma} e^{-\phi} R - \sqrt{\gamma} e^{-\phi} \gamma^{ij} \partial_i \phi \partial_j \phi + 2\sqrt{\gamma} \Delta e^{-\phi} = 0. \end{aligned} \quad (17)$$

The generating functional is assumed to be

$$S^{(2)} = \int d^3x \sqrt{\gamma} (J(\phi) R + K(\phi) \partial_i \phi \partial^i \phi). \quad (18)$$

It contains terms up to the second order in spatial gradients and is diffeomorphism invariant. We neglect the term proportional to the third diffeomorphism invariant quantity $\partial_i \partial^i \phi$, since it is possible absorb it into $K(\phi)$ [17]. Upon calculating the variational derivatives of $S^{(2)}$, substituting them into the second order Hamilton-Jacobi equation (17) and grouping together the coefficients of R , $\partial_i \phi \partial^i \phi$ and $\partial_i \partial^i \phi$, we get a system of equations for $J(\phi)$ and $K(\phi)$

$$H J + \frac{2\sqrt{3}}{\sqrt{3}-1} \frac{\partial J}{\partial \phi} H - e^{-\phi} = 0, \quad (19)$$

$$4 H \frac{\partial J}{\partial \phi} + \frac{4\sqrt{3}}{\sqrt{3}-1} H K + 2 e^{-\phi} = 0, \quad (20)$$

$$-4 H \frac{\partial^2 J}{\partial \phi^2} + H K - \frac{2\sqrt{3}}{\sqrt{3}-1} H \frac{\partial K}{\partial \phi} + e^{-\phi} = 0. \quad (21)$$

Although there are three equations for two functions $J(\phi)$ and $K(\phi)$, the system is not over-determined, since only two of them are independent. From equations (19) and (20) we get particular solutions for $J(\phi)$ and $K(\phi)$, respectively

$$J(\phi) = -\frac{\sqrt{3}(\sqrt{3}-1)}{4} e^{-(\frac{1+\sqrt{3}}{2})\phi}, \quad (22)$$

$$K(\phi) = -\frac{3(\sqrt{3}-1)}{4} e^{-(\frac{1+\sqrt{3}}{2})\phi}. \quad (23)$$

They satisfy also the third equation (21).

We are considering the case when the first order dilaton field has only time dependence, $t_0 = \text{const.}$ In this case the spatial derivatives of the dilaton are absent from field equations and we need only $J(\phi)$ for the following treatment. We can write the field equations for the dilaton (4) and for the three-metric (5) up to the second approximation as follows

$$\dot{\phi} = \sqrt{3}(\sqrt{3}+1) H - \frac{\sqrt{3}}{2} e^{\phi} J R(\gamma), \quad (24)$$

$$\dot{\gamma}_{ij} = 2 H \gamma_{ij} - \frac{1}{2} e^{\phi} J \left[(\sqrt{3}-1) \gamma_{ij} R(\gamma) + 4 R_{ij}(\gamma) \right]. \quad (25)$$

Introducing the gradient expansion explicitly and taking into account expression (12) for $H(\phi)$ we have

$$\begin{aligned}\dot{\phi}^{(0)} + \dot{\phi}^{(2)} &= (\sqrt{3} + 1) e^{(\frac{\sqrt{3}-1}{2})\phi^{(0)}} + (t_0 - t)^{-1} \phi^{(2)} \\ &+ \frac{3(\sqrt{3}-1)}{8} (t_0 - t) R(\gamma),\end{aligned}\quad (26)$$

$$\begin{aligned}\dot{\gamma}_{ij}^{(0)} + \dot{\gamma}_{ij}^{(2)} &= \frac{2}{\sqrt{3}} e^{(\frac{\sqrt{3}-1}{2})\phi^{(0)}} \gamma_{ij} + \frac{(\sqrt{3}-1)}{\sqrt{3}} (t_0 - t)^{-1} \phi^{(2)} \gamma_{ij} \\ &+ \frac{\sqrt{3}(\sqrt{3}-1)}{8} (t_0 - t) \left[(\sqrt{3}-1) \gamma_{ij} R(\gamma) + 2R_{ij}(\gamma) \right].\end{aligned}\quad (27)$$

Solving equation (26) we get for the dilaton up to the second order in spatial gradients

$$\begin{aligned}\phi(t, x^l) &= \phi^{(0)}(t) + \phi^{(2)}(t, x^l) \equiv \phi^{(0)}(t) + \delta\phi(t, x^l) \\ &= -\frac{2}{\sqrt{3}-1} \ln(t_0 - t) + \delta\phi_0(x^l) (t_0 - t)^{-1} \\ &- \frac{3\sqrt{3}}{4(11+5\sqrt{3})} (t_0 - t)^{(2+\frac{2}{\sqrt{3}})} R(k).\end{aligned}\quad (28)$$

Here $\delta\phi_0(x)$ is a space dependent integration constant and it may be interpreted as an initial dilaton perturbation. In the asymptotic past the term containing this constant is decaying and we may omit it there. However, during the inflation this term is growing (corresponding dilaton inhomogeneities are not homogenized) and becomes important near the singularity. This is in agreement with numerical treatment carried out by Chiba [10]. For the three-metric up to the second order in spatial gradients we get

$$\begin{aligned}\gamma_{ij} &= \gamma_{ij}^{(0)}(t, x^l) + \gamma_{ij}^{(2)}(t, x^l) \equiv a^2(t) k_{ij}(x^l) + a^2(t) \delta k_{ij}(t, x^l) \\ &= (t_0 - t)^{-\frac{2}{\sqrt{3}}} \{k_{ij}(x^l) + \frac{\sqrt{3}-1}{\sqrt{3}} (t_0 - t)^{-1} \delta\phi_0(x^l) k_{ij}(x^l) \\ &- \frac{3(\sqrt{3}-1)}{4(\sqrt{3}+1)} (t_0 - t)^{2+\frac{2}{\sqrt{3}}} \left[\frac{(\sqrt{3}+1)}{(5\sqrt{3}+11)} k_{ij}(x^l) R(k) + R_{ij}(k) \right]\},\end{aligned}\quad (29)$$

where $R_{ij}(k)$ and $R(k)$ are the Ricci tensor and the scalar curvature of the 3-hypersurface calculated from the seed metric $k_{ij}(x^l)$. In the second order we have incorporated the curvature as a small perturbation and expression (29) represents a non-linear evolution of the curvature and initial inhomogeneities. In expression (29) the term containing the spatial curvature becomes negligible during the superinflation as $t \rightarrow t_0$ and the initial spatial curvature and associated inhomogeneities will decay. This is in agreement with the standard result, which states, that spatial curvature

becomes negligible during any kind of inflation. In the pre-big bang cosmology the similar conclusion is obtained in [11, 13] and confirmed by numerical calculations in [9].

As we can see from expressions (28) and (29) the curvature terms generated by the seed metric become important as we go backwards in time. Since the curvature term is growing fast, we must restrict the treatment here with requirement that the second approximation (gradient terms) stays smaller than the zeroth order solution, $|S^{(0)}| > |S^{(2)}|$. This condition allows us to estimate the initial spatial curvature, and on the other hand indicates the validity of the approximation. If the initial 3-curvature and inhomogeneities are arbitrarily large, the gradient expansion cannot be used for discussing the problem of initial conditions. However, it can be used for investigating the combined restrictions on initial inhomogeneities and duration of the inflation.

6 The influence of initial inhomogeneities

Let us consider the problem of fine-tuning of initial conditions and determine the requirement for initial curvature radius for getting sufficient amount of inflation. We adopt the procedure presented by Nambu and Taruya [23] in the context of the de Sitter inflation to the pre-big bang cosmology. At each point x^l one can define a local Hubble parameter by

$$\bar{H} = \frac{\dot{\gamma}}{\gamma} = \frac{1}{6} \gamma^{ij} \dot{\gamma}_{ij}. \quad (30)$$

Using expression (25) for the three-metric one can find

$$\begin{aligned} \bar{H} &= \frac{1}{\sqrt{3}} \frac{1}{(t_0 - t)} + \frac{\sqrt{3}}{24} (4 - \sqrt{3}) (t_0 - t)^{(1 + \frac{2}{\sqrt{3}})} R(k) \\ &= H \left[1 + \frac{(4 - \sqrt{3})}{24} \frac{1}{H^2} R(\gamma^{(0)}) \right]. \end{aligned} \quad (31)$$

Here $H = (\sqrt{3}(t_0 - t))^{-1}$ is the Hubble parameter for the zeroth order solution. In the zeroth order, the Hubble horizon H^{-1} is proportional to the event horizon d_e (to be precise, $d_e = (1 + \sqrt{3})^{-1} H^{-1}$). In the second order the corresponding expression reads

$$\bar{d}_{ph} \equiv \bar{H}^{-1} = \frac{d_{ph}}{\left[1 + \frac{(4 - \sqrt{3})}{24} \frac{1}{H^2} R(\gamma^{(0)}) \right]}, \quad (32)$$

where $d_{ph} \equiv H^{-1}$. We see that the positive spatial curvature $R(\gamma^{(0)}) > 0$ reduces the horizon size compared with the zeroth order case, $\bar{d}_{ph} < d_{ph}$, and therefore favours the inflation. The negative spatial curvature $R(\gamma^{(0)}) < 0$ on the other hand increases the horizon size $\bar{d}_{ph} > d_{ph}$ and through that it is unfavorable for inflation. The situation is contrary to the case of the de Sitter inflation [23, 31].

Our conclusions are valid for small curvatures. If the curvature takes the value $R^{cr}(\gamma^{(0)}) = -\left(\frac{24}{4 - \sqrt{3}}\right) H^2$ the horizon size is infinite, $d_{ph} \rightarrow \infty$, but in this case the

gradient expansion is long ago broken because the requirement that $|S^{(0)}| > |S^{(2)}|$ is not valid.

Using the expression for \bar{H} , it is possible to calculate the number of e-folds of growth in scale factor during the superinflation including the curvature corrections

$$\begin{aligned}
\bar{N} &= \ln \left(\frac{\bar{a}(t_f)}{\bar{a}(t_i)} \right) = \int_{t_i}^{t_f} dt \bar{H} \\
&\approx \ln \left(\frac{t_0 - t_i}{t_0 - t_f} \right)^{\frac{1}{\sqrt{3}}} - \left[\frac{4 - \sqrt{3}}{16(\sqrt{3} + 1)} (t_0 - t)^{2 + \frac{2}{\sqrt{3}}} R(k) \right]_{t_i}^{t_f} \\
&\approx N_0 - p(t_0 - t_f)^2 R^f(\gamma^{(0)}) + p(t_0 - t_i)^2 R^i(\gamma^{(0)}). \tag{33}
\end{aligned}$$

Here t_i, t_f are the onset and end time of superinflation, $R^i(\gamma^{(0)})$, $R^f(\gamma^{(0)})$ are the initial and the final spatial curvatures and p is a positive numerical prefactor ($p \approx 1/20$). N_0 is the e-folding for the zeroth order solution, i.e. for spatially flat case (calculated from H). The second term in expansion (33) is suppressed if $t \rightarrow t_0$ and we can write the expression for \bar{N} as follows

$$\bar{N} \approx N_0 + p(t_0 - t_i)^2 R^i(\gamma^{(0)}) = N_0 + \frac{p}{3} \frac{1}{H_i^2} R^i(\gamma^0). \tag{34}$$

The effect of initial curvature (inhomogeneity) is the same as considered above.

Now we assume, that in the zeroth order the superinflation is long enough to satisfy the phenomenological constraints. This means that the comoving Hubble radius must decrease at least e^{60} times during the inflation [6]

$$Z = \frac{d_{cm}^i}{d_{cm}^f} = \left(\frac{t_0 - t_f}{t_0 - t_i} \right)^{1 + \frac{1}{\sqrt{3}}} > e^{60}. \tag{35}$$

In the zeroth order, N_0 and Z are related by $Z = e^{(\sqrt{3}+1)N_0}$. From this requirement we get the following constraint for the Hubble horizon H_i^{-1} at the onset moment t_i of the superinflation (we assume that $H_f^{-1} \approx \lambda_s$)

$$H_i^{-1} = e^{\sqrt{3}N_0} \lambda_s = e^{\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right) \ln Z} \lambda_s. \tag{36}$$

The second term in expression (34) has to be small with respect to N_0 and taking into account the expression (36), we get for initial curvature

$$R^i(\gamma^{(0)}) < \frac{3N_0}{p} H_i^2 < \frac{3}{\sqrt{3}+1} \frac{\ln Z}{p} e^{-\left(\frac{2\sqrt{3}}{\sqrt{3}+1}\right) \ln Z} \lambda_s^{-2}, \tag{37}$$

and for initial curvature radius

$$C_{curv}^i = \sqrt{\frac{6}{R^i(\gamma)}} = \sqrt{\frac{2p(\sqrt{3}+1)}{\ln Z}} e^{\left(\frac{\sqrt{3}}{\sqrt{3}+1}\right) \ln Z} \approx 0.07 H_i^{-1} \approx 10^{16} \lambda_s. \tag{38}$$

We see that the initial patch has to be extremely flat and also extremely homogeneous because the characteristic size of inhomogeneities must be greater than the curvature radius $L^i > C_{curv}^i$. However, the characteristic size of these inhomogeneities is only a small part of the zeroth order horizon size (cf. Gasperini [14]). This fact may be regarded as a fine-tuning of initial conditions. But it can also be regarded as a point of breakdown of the gradient expansion method for considering the problem of initial conditions.

7 Summary

In this paper we investigated the gradient expansion for the Hamilton-Jacobi equation derived from the low energy tree level effective string action. In the zeroth order we got cosmological pre-big bang solutions for the background with inhomogeneities much larger than the Hubble radius (typical horizon scale). The second order approximation includes the effect of spatial gradients. We find that metric corrections die off during the superinflation as $t \rightarrow t_0$, but dilaton corrections are growing. This means that initial classical inhomogeneities, which originate from spatial gradients of the seed metric, are smoothed out during the superinflation, but not dilaton inhomogeneities. Going backwards in time dilaton corrections become negligible, but influence of initial classical inhomogeneities of the seed metric and the initial curvature are growing. Thus the adequacy of the gradient expansion for investigating the pre-big bang superinflation decreases in the direction of the past as well as of the future (singularity). However, we can estimate the characteristic size of initial inhomogeneities and conclude that the inflating domain must be large in string units but smaller than the initial horizon.

ACKNOWLEDGEMENTS We acknowledge useful comments of J.D. Barrow and T. Chiba on an earlier version of this paper. This work was supported by the Estonian Science Foundation under grant No 3870.

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